

ONE EXAMPLE ABOUT THE RELATIONSHIP BETWEEN THE CD INEQUALITY AND CDE' INEQUALITY

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ABSTRACT. In this paper, we will give an easy example to satisfy that we can not conclude CDE' Inequality just from the CD Inequality.

1. INTRODUCTION

The curvature-dimension inequality (CD-inequality) was firstly introduced by Bakry-Emery as a substitute of the lower Ricci curvature bound of the underlying space. It was first studied on finite graphs by Lin-Yau [9][11].

Then some special cases $CD(0, \infty)$ are being studied by Liu-Peyerimhoff. They prove an optimal eigenvalue ratio estimate for finite weighted graphs satisfying the CD inequality. Also, they show taking Cartesian products to be an efficient way for constructing new weighted graphs satisfying $CD(0, \infty)$. [6][7].

In 2015, Paul-Lin-Liu-Yau prove Li-Yau type estimates for bounded and positive solutions of the heat equation on graphs, under the assumption of the curvature-dimension inequality $CDE'(0, n)$, which can be considered as a notion of curvature for graphs. So the relation between CD and CDE' is becoming more and more important for the study of the Ricci estimate on finite graphs. [2]

In 2014, 2015, Munch introduced a new version of a curvature-dimension inequality for non-negative curvature. He used this inequality to prove a logarithmic Li-Yau inequality on finite graphs. The new calculus and the new curvature-dimension inequality coincide with the common ones. In the case of graphs, they coincide in a limit. In that case, the new curvature-dimension inequality gives a more general concept of curvature on graphs. Then he showed the connection between the CDE' and the $CD\psi$ inequality. Also, he showed that the CDE' inequality implies the CD inequality. [3][4]

So can CD inequality imply CDE' ? It is a problem. In this paper, the author gives a special case to show that the hypothesis is not right in some special cases.

The paper is organized into three parts:

Chapter 1 is the introduction of the graph, the Laplacians and CD, CDE' inequalities on it.

Chapter 2 introduces some basic conclusions and lemmas in order to get the main result. These conclusions include some definitions such as local finite graph, weighted graph and the calculation of the Γ operator.

Chapter 3 is the main conclusion of this thesis.

2. GRAPHS, LAPLACIANS CD INEQUALITIES AND CDE' INEQUALITIES

Given a graph $G = (V, E)$, for an $x \in V$, if there exists another $y \in V$ that satisfies $(x, y) \in E$, we call them are neighbors, and written as $x \sim y$. If there exists an $x \in V$ satisfying $(x, x) \in E$, we call it a self-loop. For a graph $G = (V, E)$. The neighborhood and the degree of a vertex $x \in V$ are defined, respectively, as $N_x = \{y \in V : xy \in E\}$ and $d_x = |N_x|$. For notational simplicity we work with $\mu_x = 1/d_x$.

Now we will introduce some basic definitions and theorems before we get the main results.

Definition 2.1. (difference in valuations) For a function $f \in R^V$ and two vertices $x, y \in V$ denote by $f(x, y) = f(y) - f(x)$ the difference in valuations.

Definition 2.2. (locally finite graph) We call a graph G is a locally finite graph if for any $x \in V$, it satisfies $\#\{y \in V | y \sim x\} < \infty$. Moreover, it is called connected if there exists a sequence $\{x_i\}_{i=0}^n$ satisfying: $x = x_0 \sim x_1 \sim \dots \sim x_n = y$.

Definition 2.3. (Laplacians on locally finite graphs) On a locally finite graph $G = (V, E, \mu)$ the Laplacian has a form as follows:

$$\Delta f(x) = \mu_x \sum_{y \in N_x} (f(y) - f(x)), \quad \forall f \in C_0(V).$$

Definition 2.4. (gradient operator Γ) The operator Γ is defined as follows:

$$\Gamma(f, g)(x) = \frac{1}{2}(\Delta(fg) - f\Delta g - g\Delta f)(x)$$

Always we write $\Gamma(f, f)$ as $\Gamma(f)$.

Definition 2.5. (gradient operator Γ_i) The operator Γ_i is defined as follows:

$$\Gamma_0(f, g) = fg$$

$$\Gamma_{i+1}(f, g) = \frac{1}{2}(\Delta(\Gamma_i(f, g)) - \Gamma_i(f, \Delta g) - \Gamma_i(\Delta f, g))$$

Also we have $\Gamma_2(f) = \Gamma_2(f, f) = \frac{1}{2}\Delta\Gamma(f) - \Gamma(f, \Delta f)$.

Definition 2.6. ($CD(K, n)$ condition) We call a graph satisfies $CD(K, n)$ condition if for any $x \in V$, we have

$$\Gamma_2(f)(x) \geq \frac{1}{n}(\Delta f)^2(x) + K\Gamma(f)(x). \quad K \in \mathbb{R}.$$

Definition 2.7. ($CDE(K, n)$ condition) Let $f : V \rightarrow \mathbb{R}^+$ satisfy $f(x) > 0$, $\Delta f(x) < 0$. We call a graph satisfies $CDE(x, K, n)$ condition if for any $x \in V$, we have

$$\Gamma_2(f)(x) - \Gamma\left(f, \frac{\Gamma(f)}{f}\right)(x) \geq \frac{1}{n}(\Delta f)(x)^2 + K\Gamma(f)(x). \quad K \in \mathbb{R}.$$

Definition 2.8. ($CDE'(K, n)$ condition) We say that a graph G satisfies the exponential curvature dimension inequality $CDE(K, n)$ if for any positive function $f : V \rightarrow \mathbb{R}^+$ such that $\Delta f(x) < 0$, we have

$$\Gamma_2(f)(x) - \Gamma\left(f, \frac{\Gamma(f)}{f}\right)(x) \geq \frac{1}{n}f(x)^2(\Delta \log f)(x)^2 + k\Gamma(f)(x)$$

Lemma 2.9.

$$\Gamma(f)(x) = \frac{1}{2}\mu_x \sum_{y \in N_x} f(x, y)^2.$$

Here we define $N_x = \{y \in V : xy \in E\}$ and $d_x = |N_x|$. For notational simplicity we work with $\mu_x = \frac{1}{d_x}$.

Proof We have

$$\begin{aligned} \Gamma(f)(x) &= \frac{1}{2}\Delta(f^2)(x) - f(x)(\Delta f)(x) \\ &= \frac{1}{2}\mu_x \sum_{y \in N_x} (f^2)(x, y) - f(x)\mu_x \sum_{y \in N_x} f(x, y) \\ &= \frac{1}{2}\mu_x \sum_{y \in N_x} (f(x, y)(f(y) + f(x)) - 2f(x, y)f(x)) \\ &= \frac{1}{2}\mu_x \sum_{y \in N_x} f(x, y)^2. \end{aligned}$$

Lemma 2.10.

$$\Gamma_2(f)(x) = \frac{1}{2}(\Delta f)^2(x) + \mu_x \sum_{y \in N_x} \mu_y \sum_{z \in N_y} (f(y, z)^2 - \frac{1}{2}f(x, z)^2).$$

Proof We have

$$\Delta(\Gamma(f))(x) = \mu_x \sum_{y \in N_x} \Gamma(f)(x, y) = \mu_x \sum_{y \in N_x} \frac{1}{2}\mu_y \sum_{z \in N_y} (f(y, z)^2 - f(x, y)^2)$$

and

$$\begin{aligned}
\Gamma(f, \Delta f)(x) &= \frac{1}{2}(\Delta(f \cdot \Delta f)(x) - f(x) \cdot (\Delta^2 f)(x) - (\Delta f)^2(x)) \\
&= -\frac{1}{2}(\Delta f)^2(x) + \frac{1}{2}\mu_x \sum_{y \in N_x} ((f \Delta f)(x, y) - f(x)(\Delta f)(x, y)) \\
&= -\frac{1}{2}(\Delta f)^2(x) + \frac{1}{2}\mu_x \sum_{y \in N_x} f(x, y)(\Delta f)(y) \\
&= -\frac{1}{2}(\Delta f)^2(x) + \frac{1}{2}\mu_x \sum_{y \in N_x} f(x, y)\mu_y \sum_{z \in N_y} f(y, z)
\end{aligned}$$

thus

$$\begin{aligned}
\Gamma_2(f)(x) &= \frac{1}{2}\Delta(\Gamma(f))(x) - \Gamma(f, \Delta f)(x) \\
&= \frac{1}{2}(\Delta f)^2(x) + \frac{1}{2}\mu_x \sum_{y \in N_x} \mu_y \sum_{z \in N_y} (\frac{1}{2}f(y, z)^2 - \frac{1}{2}f(x, y)^2 - f(x, y)f(y, z)) \\
&= \frac{1}{2}(\Delta f)^2 + \frac{1}{2}\mu_y \sum_{y \in N_x} \mu_y \sum_{z \in N_y} (f(y, z)^2 - \frac{1}{2}f(x, z)^2)
\end{aligned}$$

Lemma 2.11. *If $\Delta f(x) < 0$ in $x \in V, CDE'(K, n)$ implies $CDE(K, n)$.*

Proof Let $f : V \rightarrow R_+$ be a positive function for which $\Delta f(x) < 0$. Since $\log f \leq s - 1$ for all positive s , we can write

$$\Delta \log f(x) = \sum_{y \sim x} (\log f(y) - \log f(x)) \leq \sum_{y \sim x} \frac{f(y) - f(x)}{f(x)} = \frac{\Delta f(x)}{f(x)} < 0.$$

Hence squaring everything reverses the above inequality and we get

$$(\Delta f(x))^2 \leq f(x)^2 (\Delta \log f(x))^2,$$

and thus $CDE(K, n)$ is satisfied

$$\Gamma_2(f)(x) \geq \frac{1}{n}f(x)^2(\Delta \log f(x))^2 + k\Gamma(f)(x) > \frac{1}{n}(\Delta f(x))^2 + k\Gamma(f)(x).$$

Lemma 2.12. *The CDE' inequality implies CD inequality.*

It was the work in [3]

3. BASIC CONCLUSION

Remark 3.1. In this section, we just concern the easiest graph: x is the initial point and its neighborhood is y , also y has another neighborhood z . So the graph consists of three points which are created from x , y and z and x and z are not connected. We give the special graph a name: EG . Also, we concern the special case in CD and CDE' inequality: $CD(0, n)$ and $CDE'(0, n)$.

Lemma 3.2. *If the EG satisfies $CD(0, m)$, then we have $m \geq 2$.*

Proof For the simplicity, we rewrite $f(x) = x, f(y) = y$ and $f(z) = z$. Then according to the lemma before, we can have the following: $\Delta f(x) = y - x, \Gamma(f)(x) = \frac{1}{2}(y - x)^2$. Also, we need to calculate the $\Gamma_2(f)$,

$$\begin{aligned}\Gamma_2(f)(x) &= \frac{1}{2}(y - x)^2 + \frac{1}{4}[(z - y)^2 - \frac{1}{2}(z - x)^2 + (x - y)^2] \\ &= \frac{1}{2}(y - x)^2 + \frac{1}{4}(z, y)^2 - \frac{1}{8}(z - x)^2 + \frac{1}{4}(x - y)^2 \\ &= \frac{1}{8}[z^2 + z(2x - 4y) + 8y^2 + 5x^2 - 12xy]\end{aligned}$$

Absolutely, we will use the knowledge of quadratic function. The minimum of $z^2 + z(2x - 4y)$ can be obtained when $z = 2y - x$. Then we take $z = 2y - x$ in the equality.

$$\begin{aligned}8y^2 + 5x^2 - 12xy + (2y - x)^2 + (2y - x)(2x - 4y) \\ = 8y^2 + 5x^2 - 12xy + 4y^2 + x^2 - 4xy + 4xy - 8y^2 - 2x^2 + 4xy \\ = 4x^2 + 4y^2 - 8xy\end{aligned}$$

So we have the estimate of $\Gamma_2(f)$

$$\Gamma_2(f) \geq \frac{1}{2}(x^2 + y^2 - 2xy) = \frac{1}{2}(x - y)^2 \geq \frac{1}{m}(y - x)^2$$

So we have $m \geq 2$.

Theorem 3.3. *If the EG satisfies $CD(0, m)$, we can take the special value in the point y , then it may not be satisfied in $CDE'(0, m)$.*

Proof Without loss of generality, we assume $x = 1$. According the theorem before, we need to find some special y that make sure the EG does not satisfy $CDE'(0, 2)$.

As before, the value of $\Gamma_2(f)$ is given

$$\Gamma_2(f) = \frac{3}{4}(x - y)^2 + \frac{1}{4}(z, y)^2 - \frac{1}{8}(z, x)^2$$

So the next work is to calculate the value of $\Gamma(f, \frac{\Gamma(f)}{f})$

$$\begin{aligned}\Gamma(f, \frac{\Gamma(f)}{f}) &= \frac{1}{2}\Delta(\Gamma(f)) - \frac{1}{2}f(x)\Delta(\frac{\Gamma(f)}{f(x)}) - \frac{1}{2}\Delta(f)\frac{\Gamma(f)}{f(x)} \\ &= \frac{1}{2}\Delta(\Gamma(f)) - \frac{1}{2}\Delta(\frac{\Gamma(f)}{f}) - \frac{1}{2}\Delta f\Gamma(f) \\ &= I_1 - I_2 - I_3.\end{aligned}$$

Then we get the value of I_1 .

$$\begin{aligned}
I_1 &= \frac{1}{2}(\Gamma(f)(y) - \Gamma(f)(x)) \\
&= \frac{1}{2} \frac{1}{4}[(y-x)^2 + (z, y)^2] - \frac{1}{2}(y-x)^2 \\
&= \frac{1}{2} \left[\frac{1}{4}(y-x)^2 + \frac{1}{4}(y-z)^2 - \frac{1}{2}(y-x)^2 \right] \\
&= \frac{1}{8}(y-z)^2 - \frac{1}{8}(y-x)^2.
\end{aligned}$$

Also the value of I_2 .

$$\begin{aligned}
I_2 &= \frac{1}{2} \frac{\Gamma(f)(y)}{y} - \frac{1}{2} \frac{\Gamma(f)(x)}{x} \\
&= \frac{1}{2} \frac{1/4[(y-x)^2 + (z-y)^2]}{y} - \frac{1}{4}(y-x)^2
\end{aligned}$$

Then the value of I_3 .

$$\begin{aligned}
I_3 &= \frac{1}{2} \Delta(f) \Gamma(f)(x) \\
&= \frac{1}{2}(y-x) \frac{1}{2}(y-x)^2 \\
&= \frac{1}{4}(y-x)^3
\end{aligned}$$

We get the following inequality according the definition of $CDE'(m, 0)$

$$\frac{3}{4}(y-x)^2 + \frac{1}{4}(y-z)^2 - \frac{1}{8}(z-x)^2 - \frac{1}{8}(y-z)^2 + \frac{1}{8}(y-x)^2 + \frac{1}{8y}(y-x)^2 + \frac{1}{8y}(y-z)^2 - \frac{1}{4}(y-x)^2 + \frac{1}{4}(y-x)^3 \geq \frac{1}{2}(lo$$

Firstly, we deal with the polynomial with z like the situation before.

$$\begin{aligned}
&\frac{1}{8}(y-z)^2 - \frac{1}{8}(z-x)^2 + \frac{1}{8y}(y-z)^2 \\
&= \frac{1}{8}(y^2 + z^2 - 2yz - z^2 - 1 + 2z) + \frac{1}{8y}(y^2 + z^2 - 2yz) \\
&= \frac{1}{8}y^2 - \frac{1}{4}yz + \frac{1}{4}z - \frac{1}{8} + \frac{1}{8}y + \frac{1}{8y}z^2 - \frac{1}{4}z \\
&= \frac{1}{8y}z^2 - \frac{1}{4}yz + \frac{1}{8}y^2 + \frac{1}{8}y - \frac{1}{8}
\end{aligned}$$

Here, we take $z = \frac{\frac{1}{4}y}{\frac{1}{4} - \frac{1}{4}y} = y^2$. Then we get

$$\frac{1}{8y}y^4 - \frac{1}{4}y^3 + \frac{1}{8}y^2 + \frac{1}{8}y - \frac{1}{8} = -\frac{1}{8}y^3 + \frac{1}{8}y^2 + \frac{1}{8}y - \frac{1}{8}.$$

Then the inequality becomes

The left of the inequality

$$\begin{aligned}
 &= \frac{1}{4}(y-1)^3 + \frac{5}{8}(y-1)^2 + \frac{1}{8y}(y-1)^2 - \frac{1}{8}y^3 + \frac{1}{8}y^2 + \frac{1}{8}y - \frac{1}{8} \\
 &= \frac{1}{4}(y^3 - 3y^2 + 3y - 1) + \frac{5}{8}y^2 + \frac{5}{8} - \frac{10}{8}y - \frac{1}{8}y^3 + \frac{1}{8}y^2 + \frac{1}{4}y - \frac{3}{8} + \frac{1}{8y} \\
 &= \frac{1}{8}y^3 + y\left(\frac{3}{4} - \frac{10}{8} + \frac{1}{8} + \frac{1}{8}\right) + \frac{1}{8y} \\
 &= \frac{1}{8}y^3 - \frac{1}{4}y + \frac{1}{8y}
 \end{aligned}$$

So the inequality becomes:

$$y^3 - 2y + \frac{1}{y} \geq 4(\log y)^2$$

. Here we distinguish y into two parts: $y > 1$ and $0 < y < 1$

I: First, we concern the situation $y > 1$, because $e^x \geq x + 1$, so we have $\log y < y - 1$

Then we need to prove

$$y^3 - 2y + \frac{1}{y} \geq 4(y-1)^2$$

It equals

$$y^3 - 4y^2 + 6y - 4 + \frac{1}{y} \geq 0$$

we set the function $h(y) = y^3 - 4y^2 + 6y - 4 + \frac{1}{y}$.

The first derivate of $h(y)$ is

$$h'(y) = 3y^2 - 8y + 6 - \frac{1}{y^2}$$

The second derivate of $h(y)$ is

$$h''(y) = 6y - 8 + 2\frac{1}{y^3}$$

Then we use the cauchy inequality to get that

$$6y - 8 + \frac{1}{y^3} = 2y + 2y + 2y + 2\frac{1}{y^3} - 8 = 2(y + y + y + \frac{1}{y^3}) - 8 \geq 2 * 4(y y y \frac{1}{y^3})^{(1/4)} - 8 = 0$$

So the function $h'(y)$ is increasing when $y > 1$.

$$h'(y) > h'(1) = 3 - 8 + 6 - 1 = 0$$

So the function $h(y)$ is increasing too.

$$h(y) \geq h(1) = 1 - 4 + 6 - 4 + 1 = 0$$

. When $y > 1$, we can get the $CDE'(0, m)$

II:Second,we concern the situation $y < 1$ as our discussion.we need to proof the following

$$y^3 - 2y + \frac{1}{y} \geq 4(\log f)^2$$

Actually, it equals that

$$y^3 - 2y + \frac{1}{y} \geq 4\left(\log \frac{1}{y}\right)^2$$

Just as discussed before,then we have to prove

$$y^3 - 2y + \frac{1}{y} \geq 4\left(\frac{1}{y} - 1\right)^2$$

Here we set $Q(y) = y^3 - 2y + \frac{1}{y} - 4\left(\frac{1}{y} - 1\right)^2 = y^3 - 2y + \frac{9}{y} - \frac{4}{y^2} - 4$. The next step is to analysis $Q(y)$

$$\begin{aligned} & y^3 - 2y + \frac{9}{y} - \frac{4}{y^2} - 4 \\ &= y^3 - y^2 + y^2 - y - y + 1 - 5 + \frac{5}{y} + \frac{4}{y} - \frac{4}{y^2} \\ &= (y - 1)(y^2 + y - 1 - \frac{5}{y} + \frac{4}{y^2}) \\ &= (y - 1)(y^2 - y + 2y - 2 + 1 - \frac{1}{y} - \frac{4}{y} + \frac{4}{y^2}) \\ &= (y - 1)^2(y + 2 + \frac{1}{y} - \frac{4}{y^2}) \\ &= (y - 1)^2(y - 1 + 3 - \frac{3}{y} + \frac{4}{y} - \frac{4}{y^2}) \\ &= (y - 1)^3(1 + \frac{3}{y} + \frac{4}{y^2}) \end{aligned}$$

Of course $1 + \frac{3}{y} + \frac{4}{y^2} > 0$,but $y - 1 < 0$,so $(y - 1)^3(1 + \frac{3}{y} + \frac{4}{y^2}) < 0$,which is contrary to what we need.Clearly,we can set $y = 0.1$,then the left is less than the right.So we get the conclusion that we can not conclude CDE' just from CD situation.

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